A generalization of the Shafer-Fink inequality

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02-03-2013

In this article we will prove some generalizations and extensions of the Shafer-Fink ([3]) double inequality for the arctangent function:

Theorem 1. *For any positive real number x,*

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan x < \frac{\pi x}{1+2\sqrt{1+x^2}}$$

holds.

Proof. Following the lines of ([2]), we consider the substitution $x = \tan\theta$, that gives the following, equivalent form of the inequality:

$$\forall \theta \in I = (0, \pi/2), \qquad \theta(\cos\theta + 2) - \pi \sin\theta < 0 < \theta(\cos\theta + 2) - 3\sin\theta.$$

If now we set

$$f_K(\theta) = (\cos\theta + 2) - K\,\frac{\sin\theta}{\theta}$$

we have:

$$\theta^2\,\frac{df_K}{d\theta} = (K - \theta^2)\sin\theta - K\theta\cos\theta.$$

Since for any $\theta \in I$ we have:

$$\frac{\theta}{\tan\theta} < 1 - \frac{\theta^2}{3} < 1 - \frac{\theta^2}{\pi},$$

$f_3(\theta)$ ed $f_\pi(\theta)$ are both non-decreasing on I, in virtue of $\frac{df_K}{d\theta} \geq 0$; moreover, $f_K'(0) = 0$ and f_K' cannot be zero on I. Since:

$$f_3(0) = 0, \quad f_3(\pi/2) > 0, \quad f_\pi(0) < 0, \quad f_\pi(\pi/2) = 0,$$

the claim follows. □

We give now a different proof of this inequality, that relies on the bisection formula for the cotangent function and the associated Weierstrass product.

From the logarithmic derivative of the Weierstrass product for the sine function we know that for any $x \in [0, \pi/2]$

$$f(x) = x \cot x = 1 - 2\sum_{k=1}^{+\infty} \frac{\zeta(2k)}{\pi^{2k}}\, x^{2k}$$

holds. Since $f(x)$ is an even function, there exists a suitable linear combination $g_1(x)$ of $f(x)$ and $f(x/2)$ that satisfies:

$$g_1(x) = A_0 f(x) + A_1 f(x/2) = 1 - \sum_{k\geq 2} C_k^{(1)}\, x^{2k}.$$

With the choices $A_0 = -\frac{1}{3}, A_1 = \frac{4}{3}$ the previous identity holds, and, for any $k \geq 2$:

$$C_k^{(1)} = \left(A_0 + \frac{A_1}{4^k}\right)\frac{\zeta(2k)}{\pi^{2k}} < 0,$$

so $g_1(x)$ is an increasing and convex function over $I = [0, \pi/2]$. From that,

$$\forall x \in I, \quad \left(-\frac{1}{3}x\cot x + \frac{2}{3}x\cot\frac{x}{2}\right) \in [g_1(0), g_1(\pi/2)] = [1, \pi/3]$$

follows. If now we consider the bisection formula for the cotangent function:

$$\cot\frac{x}{2} = \cot x + \sqrt{1+\cot^2 x}$$

we have a different proof of the Shafer-Fink inequality.

We consider now $g_2(x)$ as a linear combination of $f(x), f(x/2)$ and $f(x/4)$ such that:

$$g_2(x) = A_0 f(x) + A_1 f(x/2) + A_2 f(x/4) = 1 - \sum_{k\geq 3} C_k^{(2)}\, x^{2k}.$$

From the annihilation of the coefficient of x^2 in the RHS we deduce the constraint $A_0 + A_1\cdot\frac{1}{4} + A_2\cdot\frac{1}{16} = 0$, and from the annihilation of the coefficient of x^4 we deduce the constraint $A_0 + A_1\cdot\frac{1}{16} + A_2\cdot\frac{1}{256} = 0$. If we take $p_2(x) = A_0 + A_1 x + A_2 x^2$, such constraints translate into $p_2(1/4) = p_2(1/16) = 0$, from which:

$$p_2(x) = K_2\left(x - \frac{1}{4}\right)\left(x - \frac{1}{16}\right),$$

with $K_2 = (1-1/4)^{-1}\cdot(1-1/16)^{-1}$ in order to grant $A_0 + A_1 + A_2 = p_2(1) = 1$.
Since $C_k^{(2)} = \frac{\zeta(2k)}{\pi^{2k}} p_2(4^{-k})$, all the non-zero coefficients of the Taylor series of $g_2(x)$, except (at most) the first one, have the same sign, so $g_2(x)$ is a monotonic function over I. In particular:

$$\begin{aligned}\forall x \in I, \qquad \frac{\pi(3+8\sqrt{2})}{45} = g_2(\pi/2) \leq g_2(x) &= \frac{1}{45}\left(f(x) - 20f(x/2) + 64f(x/4)\right)\\ &= \frac{x}{45}\left(\cot x - 10\cot(x/2) + 16\cot(x/4)\right) \leq 1,\end{aligned}$$

from which we get:

$$\pi(3+8\sqrt{2}) \leq x\left(\cot x - 10\cot(x/2) + 16\cot(x/4)\right) \leq 45.$$

By using twice the bisection formula for the cotangent, we have the following strengthening of the Shafer-Fink inequality:

Theorem 2 (D'Aurizio)**.** *For any positive real number x*

$$\pi(3+8\sqrt{2})\cdot f(x) < \arctan x < 45\cdot f(x)$$

holds, where:

$$f(x) = \frac{x}{7+6\sqrt{1+x^2}+16\sqrt{2}\sqrt{1+x^2+\sqrt{1+x^2}}}.$$

The same approach leads to an arbitrary strengthening of the Shafer-Fink inequality:

Theorem 3 (D'Aurizio)**.** *For any positive real number x and for any positive natural number n, once defined:*

$$f(x) = x\cot x = 1 - 2\sum_{k=1}^{+\infty}\frac{\zeta(2k)}{\pi^{2k}}\,x^{2k},$$

$$p_n(x) = \prod_{k=1}^{n}\frac{(4^k x-1)}{(4^k-1)} = A_0 + A_1 x + \ldots + A_n x^n,$$

$$g_n(x) = \sum_{k=0}^{n} A_k\, f(2^{-k}x) = x\sum_{k=0}^{n}\frac{A_k}{2^k}\,\cot(2^{-k}x),$$

$$e_j(x_1,\ldots,x_k) = \sum_{sym} x_1\cdot\ldots\cdot x_j,$$

$$L_0(x) = 1, \qquad L_{n+1}(x) = L_n(x) + \sqrt{x^2+L_n(x)^2},$$

we have:

$$K_{low}\cdot a_n(x) < \arctan(x) < K_{high}\cdot a_n(x),$$

where $K_{low} = \min(g_n(0), g_n(\pi/2))$, $K_{high} = \max(g_n(0), g_n(\pi/2))$ and:

$$a_n(x) = x\cdot\left(\sum_{j=0}^{n}(-1)^{n-j}\cdot L_j(x)\cdot 2^j\cdot e_j(1,4,\ldots,4^{n-1})\right)^{-1}.$$

Moreover, $K_{high} - K_{low} < \frac{1}{4^n}$.

Proof. By taking

$$p_n(x) = \prod_{k=1}^{n}\frac{(4^k x-1)}{(4^k-1)} = A_0 + A_1 x + \ldots + A_n x^n$$

we have $p_n(1) = 1$ and $p_n(4^{-j}) = 0$ for every $j \in [1,n]$. In particular, the Taylor series of

$$g_n(x) = \sum_{k=0}^{n} A_k\, f(2^{-k}x) = x\sum_{k=0}^{n}\frac{A_k}{2^k}\,\cot(2^{-k}x).$$

is equal to:

$$1 - 2\sum_{k=1}^{+\infty}\frac{\zeta(2k)p_n(4^{-k})}{\pi^{2k}}\,x^{2k} = 1 - 2\sum_{k>n} C_n^{(k)}\,x^{2k},$$

and all the $C_n^{(k)}$ with $k > n$ have the same sign, so $g_n(x)$ is monotonic over $[0, \pi/2]$, with $g_n(0) = 1$. In particular, we have:

$$\forall x \in [0, \pi/2], \qquad x \cdot \sum_{j=0}^{n} (-1)^{n-j} \cot\left(\frac{x}{2^j}\right) 2^j \, e_j(1, 4, \ldots, 4^{n-1}) \leq \prod_{k=1}^{n} (4^k - 1),$$

where e_j is the j-th elementary symmetric function. Since for any $m > n$ we have $|p_n(4^{-m})| < 1$,

$$|g_n(\pi/2) - g_n(0)| \leq \sum_{k>n} \frac{\zeta(2k)}{4^k} < \frac{1}{4^n}.$$

holds. □

We give now another upper bound for the arctangent function that does not belong to the last family of inequalities, but that strenghtens the inequality $\arctan x < \frac{\pi x}{1+2\sqrt{1+x^2}}$, too.

Theorem 4. *For any positive real number x*

$$\arctan x < \frac{\pi x}{\frac{4}{\pi} + \sqrt{2}\sqrt{1 + x^2 + x\sqrt{1+x^2}}}$$

holds.

Proof. By using the substitution $x = \tan\theta$, it is sufficient to prove that for any $\theta \in I = [0, \pi/2]$ we have:

$$\theta \leq \frac{\pi \sin\theta}{\frac{4}{\pi}\cos\theta + \sqrt{2 + 2\sin\theta}},$$

that is also equivalent, up to the change of variable $\theta = \pi/2 - \phi$, to the inequality:

$$\frac{\pi}{2} - \phi \leq \frac{\pi \cos\phi}{\frac{4}{\pi}\sin\phi + 2\cos(\phi/2)},$$

or the inequality:

$$\frac{\cos\phi}{1 - \frac{2\phi}{\pi}} \geq \cos(\phi/2)\left(\frac{4}{\pi}\sin(\phi/2) + 1\right).$$

In order to prove the latter it is sufficient to prove:

$$\frac{\cos\phi}{1 - \frac{2\phi}{\pi}} \geq \cos(\phi/2)\left(1 + \frac{2\phi}{\pi}\right),$$

or:

$$\frac{\cos\phi}{1 - \frac{4\phi^2}{\pi^2}} \geq \cos(\phi/2).$$

By considering the Weierstrass product for the cosine function we may rewrite the last line in the form:

$$\prod_{k=1}^{+\infty}\left(1 - \frac{4x^2}{(2k+1)^2\pi^2}\right) \geq \prod_{k=1}^{+\infty}\left(1 - \frac{x^2}{(2k-1)^2\pi^2}\right).$$

By considering the Taylor series of the logarithm of both sides, we simply have to prove:

$$\forall m \in \mathbb{N}_0, \qquad (4^m - 1)\zeta(2m) - 4^m - (1 - 4^{-m})\zeta(2m) \leq 0,$$

that is a consequence of:

$$\forall m \in \mathbb{N}_0, \qquad \zeta(2m) \leq \frac{4^m+1}{4^m-1},$$

implied by:

$$\forall m \in \mathbb{N}_0, \qquad (4^m-1)(\zeta(2m)-1) \leq 2.$$

An upper bound for the LHS is the series:

$$1+\sum_{k=1}^{+\infty}\left(\frac{4}{(2k+1)^2}\right)^m,$$

whose value decreases as m increases; so we have:

$$(4^m-1)(\zeta(2m)-1) \leq 1+\sum_{k=1}^{+\infty}\frac{4}{(2k+1)^2} = 3\zeta(2)-3,$$

and the RHS is less than 2 since $\pi^2 < 10$ holds. □

Now we make a step back into the general setting of double inequalities for the arctangent function.

Lemma 1. *If $f(u), g(u)$ are a couple of real functions such that, for any $u \in [0,1]$,*

$$f(u) \leq \arctan u \leq g(u)$$

holds, then:

$$2\cdot f\left(\frac{x}{1+\sqrt{1+x^2}}\right) \leq \arctan x \leq 2\cdot g\left(\frac{x}{1+\sqrt{1+x^2}}\right)$$

holds for any $x \in \mathbb{R}^+$.

Proof. In virtue of the angle bisector theorem,

$$\arctan t = 2\arctan\left(\frac{t}{1+\sqrt{1+t^2}}\right)$$

for any $t \geq 0$, so if the first inequality holds for any $\theta = \arctan u$ in the range $[0,\pi/4]$, the second inequality holds for any $\theta = \arctan x$ in the range $[0,\pi/2]$. □

The last lemma gives a third way to prove the Shafer-Fink inequality. By direct inspection of the Taylor series of $\frac{\arctan u}{u}$, it is easy to show that $(3+u^2)\frac{\arctan u}{u}$ is an increasing function over $[0,1]$, so:

$$\frac{3u}{3+u^2} \leq \arctan u \leq \frac{\pi u}{3+u^2},$$

and it is sufficient to use the substitution $u = \frac{x}{1+\sqrt{1+x^2}}$ to give another proof of the Shafer-Fink inequality.

Lemma 2. *If an approximation $f(u)$ of the arctangent function satisfies:*

$$\|f(u)-\arctan(u)\|_{\mathbb{R}^+} = \sup_{u\in\mathbb{R}^+}|f(u)-\arctan(u)| = C_\infty,$$

then

$$\left\|2\cdot f\left(\frac{u}{1+\sqrt{1+u^2}}\right)-\arctan(u)\right\|_{\mathbb{R}^+} = 2\cdot\|f(u)-\arctan(u)\|_{(0,1)} = 2\cdot C_1,$$

and, for any $t \in (0,1)$*,*

$$\left\| 2 \cdot f\left(\frac{u}{1+\sqrt{1+u^2}}\right) - \arctan(u) \right\|_{(0,t)} = 2 \cdot \| f(u) - \arctan(u) \|_{\left(0, \frac{2t}{1-t^2}\right)}.$$

This simple consequence of the previous lemma tell us the fact that any algebraic approximation of the arctangent function in a right neighbourhood of zero can be "lifted" to an algebraic approximation over the whole $\mathbb{R}^+$, through the iteration of the map

$$f(u) \quad \longrightarrow \quad 2 \cdot f\left(\frac{u}{1+\sqrt{1+u^2}}\right).$$

For example, if we consider the Lagrange interpolation polynomial for the arctangent function with respect to the points $(0, \tan(\pi/8) = \sqrt{2}-1, \tan(\pi/4) = 1)$

$$p(x) = \frac{\pi}{4} \cdot \frac{x(x-\sqrt{2}+1)}{2-\sqrt{2}} + \frac{\pi}{8} \cdot \frac{x(x-1)}{(\sqrt{2}-1)(\sqrt{2}-2)},$$

we have

$$\| p(x) - \arctan x \|_{(0,1)} < \frac{1}{230},$$

so, by considering $2 \cdot p\left(\frac{x}{1+\sqrt{1+x^2}}\right)$:

Theorem 5. *For any non negative real number* x*, the absolute difference between* $\arctan(x)$ *and*

$$\frac{\pi x \left((4+\sqrt{2}) \left(1+\sqrt{1+x^2}\right) - \sqrt{2}\, x \right)}{8 \left(1+\sqrt{1+x^2}\right)^2}$$

is less than $\frac{1}{115}$*.*

Another way to produce really effective approximation is to use the Chebyshev expansion for the arctangent function:

Lemma 3. *The sequence of functions:*

$$f_n(x) = 2 \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)(1+\sqrt{2})^{2k+1}}\, T_{2k+1}(x),$$

where $T_k(x)$ *is the* k*-th Chebyshev polynomial of the first kind, gives a uniform approximation of the arctangent function over the interval* $[0,1]$*:*

$$\| \arctan x - f_n(x) \|_{[0,1]} \leq \frac{1}{(1+\sqrt{2})^{2n+3}}.$$

Moreover,

$$\arctan(mx) = 2 \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)} \left(\frac{m}{1+\sqrt{1+m^2}} \right)^{2k+1} T_{2k+1}(x)$$

holds for any $x \in (-1,1)$ *and for any* $m \in \mathbb{N}_0$*.*

Theorem 6. *For any $n \in \mathbb{N}_0$ and for any $x \in \mathbb{R}$*

$$\left|\arctan x - 4\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)(1+\sqrt{2})^{2k+1}}\,T_{2k+1}\left(\frac{x}{1+\sqrt{1+x^2}}\right)\right| \leq \frac{1}{\left(3+2\sqrt{2}\right)^n}.$$

Still another way is to use the continued fraction representation for the arctangent funtion:

$$\arctan z = \cfrac{z}{1+\cfrac{z^2}{3+\cfrac{4z^2}{5+\cfrac{9z^2}{7+\cfrac{16z^2}{9+\cfrac{25z^2}{11+\ldots}}}}}},$$

from which we get a sequence of approximations for $\arctan x$ over $[0,1]$:

$$\begin{cases} K_1(x) &= \dfrac{x}{1+x^2/3}, \\ K_2(x) &= \dfrac{x}{1+x^2/(3+4x^2/5)} &= \dfrac{x(15+4x^2)}{15+9x^2}, \\ K_3(x) &= \dfrac{x}{1+x^2/(3+4x^2/(5+9x^2/7))} &= \dfrac{5x\left(21+11x^2\right)}{105+90x^2+9x^4} \\ \ldots \end{cases}$$

that satisfy:

$$\|\arctan x - K_n(x)\|_{[0,1]} \leq \frac{1}{2\cdot 4^n},$$

so:

$$\left\|\arctan x - K_n\left(\frac{x}{1+\sqrt{1+x^2}}\right)\right\|_{\mathbb{R}} \leq \frac{1}{4^n},$$

with an error term that is the same achieved by $a_n(x)$, defined as in Theorem (3).

By using the Taylor series for the arctangent function with respect to the point $x=1$ one has:

$$\arctan x = \frac{\pi}{4} - \sum_{j=0}^{+\infty}\left(-\frac{(1-x)^4}{4}\right)^j \cdot \left(\frac{(1-x)}{2(4j+1)} + \frac{(1-x)^2}{2(4j+2)} + \frac{(1-x)^3}{4(4j+3)}\right).$$

By plugging in $x = 2/3$ we have:

$$\arctan\frac{1}{5} = \sum_{j=0}^{+\infty}\left(-\frac{1}{324}\right)^j \cdot \left(\frac{1}{6(4j+1)} + \frac{1}{18(4j+2)} + \frac{1}{108(4j+3)}\right),$$

and by plugging in $x = 119/120$ we have:

$$\arctan\frac{1}{239} = \sum_{j=0}^{+\infty}\left(-\frac{1}{829440000}\right)^j \cdot \left(\frac{1}{240(4j+1)} + \frac{1}{28800(4j+2)} + \frac{1}{6912000(4j+3)}\right).$$

The Machin Formula

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} + \arctan\frac{1}{239}$$

give us the possibility to exhibit a good approximation for π:

$$\pi = 8\sum_{j=0}^{+\infty}\left(-\frac{1}{324}\right)^j\cdot\left(\frac{1}{3(4j+1)}+\frac{1}{9(4j+2)}+\frac{1}{54(4j+3)}\right)+$$
$$+\sum_{j=0}^{+\infty}\left(-\frac{1}{829440000}\right)^j\cdot\left(\frac{1}{60(4j+1)}+\frac{1}{7200(4j+2)}+\frac{1}{1728000(4j+3)}\right).$$

In the same fashion, we have that:

$$\arctan\frac{1}{2z-1}=\sum_{j=0}^{+\infty}\left(-\frac{1}{4z^4}\right)^j\cdot\left(\frac{1}{2z(4j+1)}+\frac{1}{2z^2(4j+2)}+\frac{1}{4z^3(4j+3)}\right)$$

holds for any $z \geq 1$, and the truncated series gives a better and better approximation as z goes to infinity. By a change of variable, the same is true for:

$$\arctan\frac{1}{t}=\sum_{j=0}^{+\infty}\left(-\frac{4}{(t+1)^4}\right)^j\cdot\left(\frac{1}{(t+1)(4j+1)}+\frac{2}{(t+1)^2(4j+2)}+\frac{2}{(t+1)^3(4j+3)}\right),$$

and:

$$\arctan u=\sum_{j=0}^{+\infty}\left(-\frac{4u^4}{(u+1)^4}\right)^j\cdot\left(\frac{u}{(u+1)(4j+1)}+\frac{2u^2}{(u+1)^2(4j+2)}+\frac{2u^3}{(u+1)^3(4j+3)}\right)$$

holds for any $u \in [0,1]$. By taking:

$$s_n(u)=\sum_{j=0}^{n}\left(-\frac{4u^4}{(u+1)^4}\right)^j\cdot\left(\frac{u}{(u+1)(4j+1)}+\frac{2u^2}{(u+1)^2(4j+2)}+\frac{2u^3}{(u+1)^3(4j+3)}\right)$$

we have that:

$$|\arctan u - s_n(u)| \leq \left(\frac{\sqrt{2}\,u}{u+1}\right)^{4n}$$

for any $u \in [0,1]$, with s_n being an upper bound for $\arctan u$ over $[0,1]$ for any even n and a lower bound for any odd n. If we consider:

$$t_n(u)=\frac{\pi}{4}-s_n\left(\frac{1-u}{1+u}\right)=\frac{\pi}{4}-\sum_{j=0}^{n}\left(-\frac{(1-u)^4}{4}\right)^j\cdot\left(\frac{1-u}{2(4j+1)}+\frac{(1-u)^2}{2(4j+2)}+\frac{(1-u)^3}{4(4j+3)}\right),$$

then t_n is a lower/upper bound for the arctangent function over $[0,1]$ if and only if s_n is a lower/upper bound, and:

$$|\arctan u - t_n(u)| \leq \left(\frac{1-u}{\sqrt{2}}\right)^{4n}$$

holds. Any convex combination of s_n and t_n is still a lower/upper bound - by taking:

$$w_n(u)=\frac{u^{4n+4}\cdot t_n(u)+(1-u)^{4n+4}\cdot s_n(u)}{u^{4n+4}+(1-u)^{4n+4}}$$

we can perform a reduction of the uniform error, since:

$$|w_n(u)-\arctan u| \leq \frac{1}{20^n}$$

and the error function goes very fast to zero when u approaches 0 or 1. This gives that

$$w_n\left(\frac{u}{1+\sqrt{1+u^2}}\right)$$

is an especially good lower/upper bound for the arctangent function when u is close to 0 or much bigger than 1, achieving the same uniform error term with respect to the generalized Shafer-Fink inequality or the continued fraction expansion.

References

[1] A.M. Fink: Two inequalities, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. 6 (1995), 48–49. (`http://pefmath.etf.bg.ac.yu/`)

[2] Feng Qi, Shi-Qin Zhang, Bai-Ni Guo: Sharpening and generalizations of Shafer's inequality for the arc tangent function, Journal of Inequalities and Applications 2009 (2009).

[3] R. E. Shafer, E 1867, Amer. Math. Monthly 73 (1966), no. 3, 309.

[4] D. S. Mitrinović, Elementary Inequalities, Groningen, 1964.